ECE 604, Lecture 26

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Contents

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1 Radiation Field or Far-Field Approximation

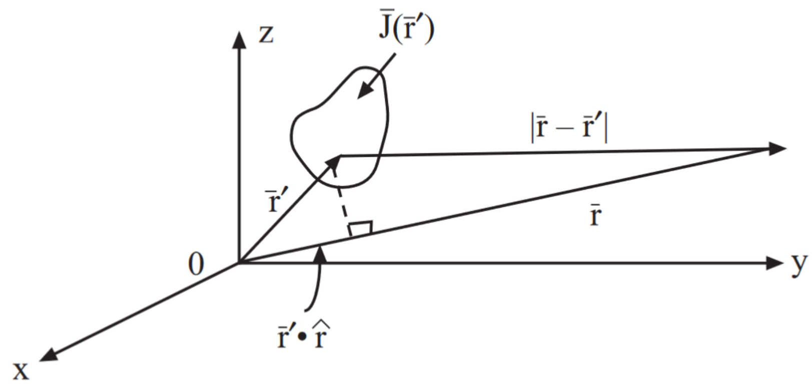

Figure 1:

In the previous lecture, we have derived the relation of the vector and scalar potentials to the sources $\mathbf{J}$ and ϱ. They are given by

$$\mathbf{A}(\mathbf{r}) = \mu \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{1.1}$$

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon} \iiint_V d\mathbf{r}' \varrho(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{1.2}$$

The integrals in (1.1) and (1.2) are normally untenable, but when the observation point is far from the source, approximation to the integral can be made giving it a nice physical interpretation.

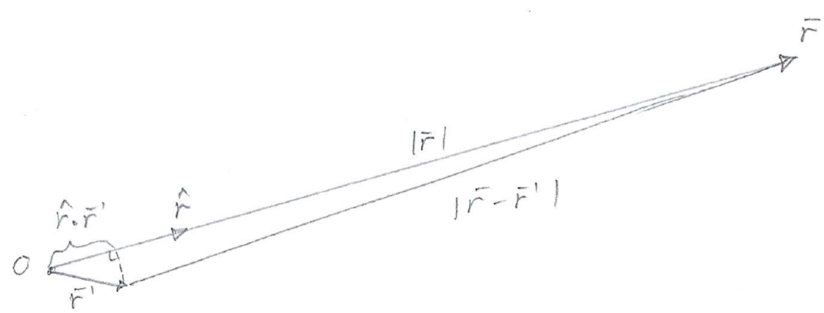

Figure 2:

1.1 Far-Field Approximation

When $|\mathbf{r}| \gg |\mathbf{r}'|$, then $|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}'\cdot\hat{r}$, where $r = |\mathbf{r}|$ and $r' = |\mathbf{r}'|$. This approximation can be shown algebraically or by geometrical argument as shown in Figure 2. Thus (1.1) above becomes

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu}{4\pi}\iiint_V d\mathbf{r}'\,\frac{\mu\mathbf{J}(\mathbf{r}')}{r-\mathbf{r}'\cdot\hat{r}}e^{-j\beta r+j\beta\mathbf{r}'\cdot\hat{r}} \approx \frac{\mu e^{-j\beta r}}{4\pi r}\iiint_V d\mathbf{r}'\mathbf{J}(\mathbf{r}')e^{j\beta\mathbf{r}'\cdot\hat{r}} \tag{1.3}$$

In the above we have made used of that $1/(1-\Delta) \approx 1$ when Δ is small, but $e^{j\beta\Delta} \neq 1$, unless $j\beta\Delta \ll 1$. Hence, we keep the exponential term in (1.3) but simplify the denominator to arrive at the last expression above.

If we let $\boldsymbol{\beta} = \beta\hat{r}$, and $\mathbf{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z'$, then

$$e^{j\beta\mathbf{r}'\cdot\hat{r}} = e^{j\boldsymbol{\beta}\cdot\mathbf{r}'} = e^{j\beta_x x'+j\beta_y y'+j\beta_z z'} \tag{1.4}$$

Therefore (1.3) resembles a 3D Fourier transform integral, namely

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu e^{-j\beta r}}{4\pi r}\iiint_V d\mathbf{r}'\mathbf{J}(\mathbf{r}')e^{j\boldsymbol{\beta}\cdot\mathbf{r}'} \tag{1.5}$$

and (1.5) can be rewritten as

$$\mathbf{A}(\mathbf{r}) \cong \frac{\mu e^{-j\beta r}}{4\pi r}\mathbf{F}(\boldsymbol{\beta}) \tag{1.6}$$

where

$$\mathbf{F}(\boldsymbol{\beta}) = \iiint_V d\mathbf{r}'\mathbf{J}(\mathbf{r}')e^{j\boldsymbol{\beta}\cdot\mathbf{r}'} \tag{1.7}$$

is the 3D Fourier transform of $\mathbf{J}(\mathbf{r}')$ with $\boldsymbol{\beta} = \hat{r}\beta$.

It is to be noted that this is not a normal 3D Fourier transform because $|\boldsymbol{\beta}|^2 = {\beta_x}^2 + {\beta_y}^2 + {\beta_z}^2 = \beta^2$. In other words, the length of the vector $\boldsymbol{\beta}$ is fixed to be β. It is the 3D "Fourier transform" of the current source $\mathbf{J}(\mathbf{r}')$ with Fourier variables, β_x, β_y, β_z lying on a sphere of radius β and $\boldsymbol{\beta} = \beta\hat{r}$. This spherical surface in the Fourier space is also called the Ewald's sphere. In a normal 3D Fourier transform, ${\beta_x}^2 + {\beta_y}^2 + {\beta_z}^2$ has values ranging from zero to infinity.

1.2 Locally Plane Wave Approximation

We can write $\hat{r}$ or $\boldsymbol{\beta}$ in terms of direction cosines in spherical coordinates or that

$$\hat{r} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta \tag{1.8}$$

Hence

$$\mathbf{F}(\boldsymbol{\beta}) = \mathbf{F}(\beta\hat{r}) = \mathbf{F}(\beta,\theta,\phi) \tag{1.9}$$

Also in (1.6), when $r \gg \mathbf{r}' \cdot \hat{r}$, $e^{-j\beta r}$ is now a rapidly varying function of r while, $\mathbf{F}(\boldsymbol{\beta})$ is only a slowly varying function of θ and ϕ, the observation angles. In other words, the prefactor in (1.6), $\exp(-j\beta r)/r$, can be thought of as resembling a spherical wave. Hence, if one follows a ray of this spherical wave and moves in the r direction, the predominant variation of the field is due to $e^{-j\beta r}$, whereas the direction of the vector $\boldsymbol{\beta}$ changes little, and hence $\mathbf{F}(\boldsymbol{\beta})$ changes little.

The above shows that in the far field, the wave radiated by a finite source resembles a spherical wave. Moreover, a spherical wave resembles a plane wave when one is sufficiently far from the source. Hence, we can write $e^{-j\beta r} = e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ where $\boldsymbol{\beta} = \hat{r}\beta$ and $\mathbf{r} = \hat{r}r$ so that a spherical wave resembles a plane wave locally. This phenomenon is shown in Figure 3.

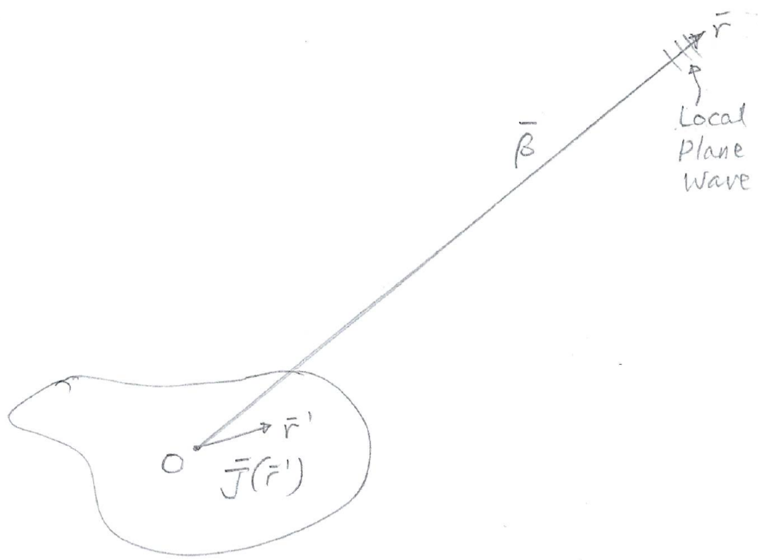

Figure 3: A spherical wave emanating from a source becomes locally a plane wave in the far field.

Then, it is clear that with the plane-wave approximation, $\nabla \to -j\boldsymbol{\beta} = -j\beta\hat{r}$, and

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \approx -j\frac{\beta}{\mu}\hat{r} \times (\hat{\theta}A_\theta + \hat{\phi}A_\phi) = j\frac{\beta}{\mu}(\hat{\theta}A_\phi - \hat{\phi}A_\theta) \tag{1.10}$$

Similarly

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla \times \mathbf{H} \cong -j\omega(\hat{\theta}A_\theta + \hat{\phi}A_\phi) \tag{1.11}$$

Notice that $\boldsymbol{\beta} = \beta\hat{r}$ is orthogonal to $\mathbf{E}$ and $\mathbf{H}$ in the far field, a property of a plane wave. Moreover, there are more than one way to derive the electric field $\mathbf{E}$. Using (1.10) for the magnetic field, the electric field can also be written as

$$\mathbf{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{A} \tag{1.12}$$

Using the formula for the double-curl operator, the above can be rewritten as

$$\mathbf{E} = \frac{1}{j\omega\mu\varepsilon}\left(-\nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A}\right) = \frac{1}{j\omega\mu\varepsilon}\left(-\boldsymbol{\beta}\boldsymbol{\beta} + \beta^2\overline{\mathbf{I}}\right)\cdot\mathbf{A} \tag{1.13}$$

where we have used that $\nabla^2\mathbf{A} = -\beta^2\mathbf{A}$. Alternatively, we can rewrite the above as

$$\mathbf{E} = -j\omega\left(-\hat{\beta}\hat{\beta} + \overline{\mathbf{I}}\right)\cdot\mathbf{A} = -j\omega\left(-\hat{r}\hat{r} + \overline{\mathbf{I}}\right)\cdot\mathbf{A} \tag{1.14}$$

Since $\overline{\mathbf{I}} = \hat{r}\hat{r} + \hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}$, then the above becomes

$$\mathbf{E} = -j\omega\left(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}\right)\cdot\mathbf{A} = -j\omega(\hat{\theta}A_\theta + \hat{\phi}A_\phi) \tag{1.15}$$

which is the same as previously derived. It also shows that the electric field is transverse to the $\boldsymbol{\beta}$ vector. We can also arrive at the above by lettering $\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$, and using the appropriate formula for the scalar potential.

Furthermore, it can be shown that in the far field, using the plane-wave approximation,

$$|\mathbf{E}|/|\mathbf{H}| \approx \eta \tag{1.16}$$

where η is the intrinsic impedance of free space, which is a property of a plane wave. Moreover, one can show that the time average Poynting's vector in the far field is

$$\langle\mathbf{S}\rangle \approx \frac{1}{2\eta}|\mathbf{E}|^2\hat{r} \tag{1.17}$$

which resembles also the property of a plane wave. Since the radiated field is a spherical wave, the Poynting's vector is radial. Therefore,

$$\langle\mathbf{S}\rangle = \hat{r}S_r(\theta,\phi) \tag{1.18}$$

The plot of $|\mathbf{E}(\theta,\phi)|$ is termed the far-field pattern or the radiation pattern of an antenna or the source, while the plot of $|\mathbf{E}(\theta,\phi)|^2$ is its far-field power pattern.

1.3 Directive Gain Pattern Revisited

Once the far-field power pattern S_r is known, the total power radiated by the antenna can be found by

$$P_T = \int_0^\pi \int_0^{2\pi} r^2 \sin\theta d\theta d\phi S_r(\theta,\phi) \tag{1.19}$$

The above evaluates to a constant independent of r due to energy conservation. Now assume that this same antenna is radiating isotropically in all directions, then the average power density of this fictitious isotropic radiator as $r \to \infty$ is

$$S_{\mathrm{av}} = \frac{P_T}{4\pi r^2} \tag{1.20}$$

A dimensionless directive gain pattern can be defined such that

$$G(\theta, \phi) = \frac{S_r(\theta, \phi)}{S_{\mathrm{av}}} = \frac{4\pi r^2 S_r(\theta, \phi)}{P_T} \tag{1.21}$$

The above function is independent of r in the far field since $S_r \sim 1/r^2$ in the far field. The directivity of an antenna $D = \max(G(\theta, \phi))$, is the maximum value of the directive gain. It is to be noted that by its mere definition,

$$\int d\Omega G(\theta, \phi) = 4\pi \tag{1.22}$$

where $\int d\Omega = \int_0^\pi \int_0^{2\pi} \sin\theta d\theta d\phi$. It is seen that since the directive gain pattern is normalized, when the radiation power is directed to the main lobe of the antenna, the corresponding side lobes and back lobes will be diminished.

An antenna also has an effective area or aperture, such that if a plane wave carrying power density denoted by S_{inc} impinges on the antenna, then the power received by the antenna, P_{received} is given by

$$P_{\mathrm{received}} = S_{\mathrm{inc}} A_e \tag{1.23}$$

A wonderful relationship exists between the directive gain pattern $G(\theta, \phi)$ and the effective aperture, namely that[1]

$$A_e = \frac{\lambda^2}{4\pi} G(\theta, \phi) \tag{1.24}$$

Therefore, the effective aperture of an antenna is also direction dependent.

[1] The proof of this formula is beyond the scope of this course.

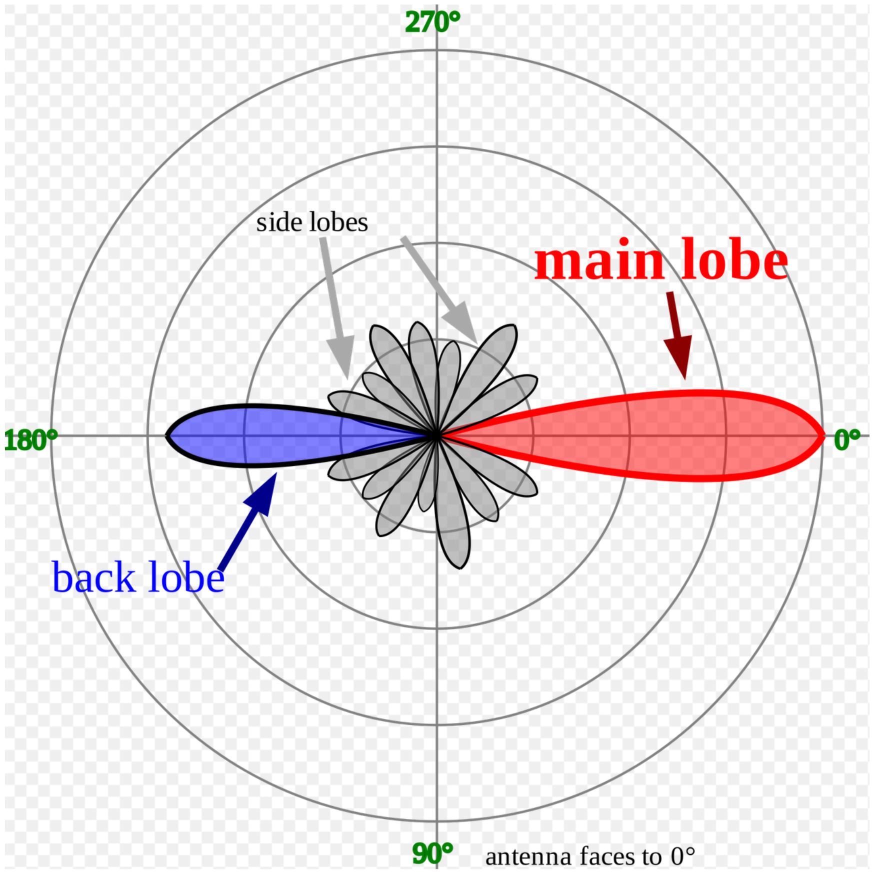

Figure 4: